Reparameterized Tensor Ring Functional Decomposition for Multi-Dimensional Data Recovery

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CVPR 2026

Definition (Tensor Ring Decomposition)

Tensor Ring (TR) decomposition represents d-th order tensor $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ using 3rd-order cores $\mathcal{G}^{(k)} \in \mathbb{R}^{r_k \times n_k \times r_{k+1}}$:

$$\mathcal{X}_{\mathbf{v}} = \text{trace}\left(\mathcal{G}^{(1)}_{:v_1:} \mathcal{G}^{(2)}_{:v_2:} \cdots \mathcal{G}^{(d)}_{:v_d:}\right).$$

- Multi-dimensional data often exhibit **low-rank** structures.
- **Tensor Ring (TR) decomposition** is a powerful low-rank model.
- **Limitation:** Traditional TR formulations are **discrete**, which limits their applicability to **continuous signals.**

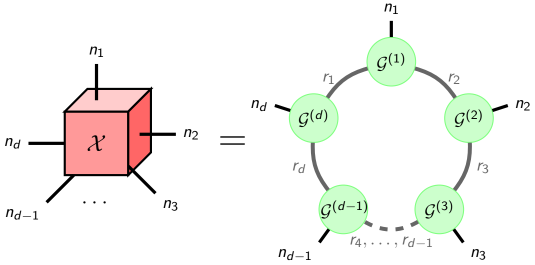


Figure: Illustration of TR decomposition.

Tensor Functional Decomposition

- Recent works have extended tensor decomposition models to the **continuous domain**.
- Tensor factors are parameterized as coordinate-dependent functions via **Implicit Neural Representations (INRs)**.

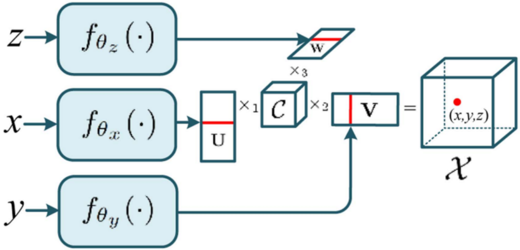


Figure: Tensor functional decomposition based on Tucker decomposition (Luo et al., TPAMI 2023).

Y. Luo, X. Zhao, Z. Li, M. K. Ng, and D. Meng, "Low-rank tensor function representation for multi-dimensional data recovery," *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 46(5):3351–3369, 2023.

Tensor Ring Functional Decomposition (TRFD)

We first introduce a continuous **Tensor Ring Functional Decomposition (TRFD)**.

- Given a coordinate vector $\mathbf{v} = [v_1, \ldots, v_d]^T$, the shared embedding for the k-th dimension is:

$$\mathbf{z}_k = \sin(\omega_0(\mathbf{w}v_k + \mathbf{b})) \in \mathbb{R}^h, \quad k = 1, \ldots, d.$$

- Each TR factor slice is then generated by an MLP $f_{\theta_k}(\cdot)$:

$$\mathcal{G}^{(k)}_{:v_k:} = f_{\theta_k}(\mathbf{z}_k) = g_{\theta_k}(v_k) \in \mathbb{R}^{r_k \times r_{k+1}}.$$

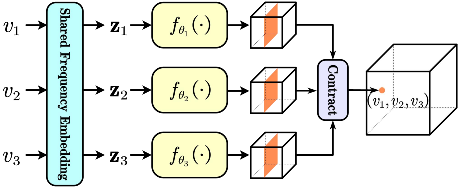


Figure: Illustration of proposed TRFD.

- The frequency characteristics of TR factors are **inherited** by the reconstructed tensor.
- The **spectral bias** of INRs makes it challenging to learn high-frequency TR factors.

Theorem 1 (Frequency Analysis of TR Factors)

Let $\mathcal{X} = \Phi(\mathcal{G}^{(1)}, \ldots, \mathcal{G}^{(d)})$. Suppose the mode-2 frequency components of $\mathcal{G}^{(k)}$ satisfy

$$\|(\mathfrak{F}_2[\mathcal{G}^{(k)}])_{:\omega_k:}\|_\infty \leq \epsilon, \quad \forall|\omega_k| > \Omega_k, k = 1, \ldots, d.$$

Then the reconstructed tensor $\mathcal{X}$ exhibits:

$$\|(\mathfrak{F}_k[\mathcal{X}])_{:\omega_k:}\|_\infty \leq c_k\epsilon, \quad \forall|\omega_k| > \Omega_k, k = 1, \ldots, d.$$

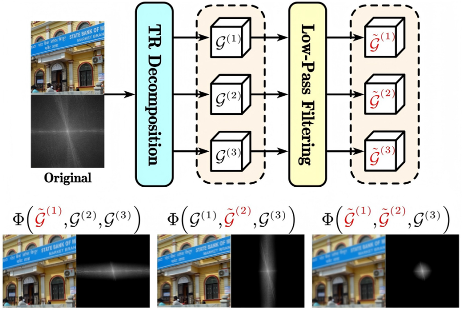


Figure: Frequency Analysis of TR Factors.

Reparameterized TRFD (RepTRFD)

To improve training dynamics, we **reparameterize each TR factor** as:

$$\mathcal{G}^{(k)} = \mathcal{C}^{(k)} \times_3 \mathbf{B}^{(k)}.$$

- $\mathcal{C}^{(k)} \in \mathbb{R}^{r_k \times n_k \times R_{k+1}}$ is the trainable latent tensor with $R_{k+1} = \beta r_{k+1}$ $(\beta \geq 1)$.
- $\mathbf{B}^{(k)} \in \mathbb{R}^{r_{k+1} \times R_{k+1}}$ is the fixed basis.

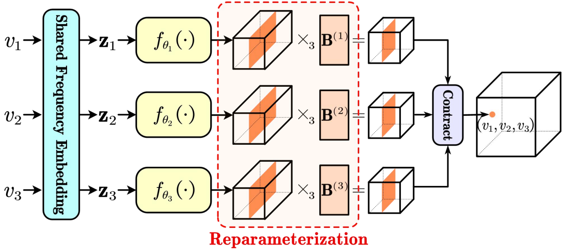


Figure: Illustration of proposed RepTRFD.

Theoretical Guarantee: Training Dynamics

Why does this reparameterization help?

Theorem 2 (Gradient Amplification)

Consider $\mathcal{G} = \mathcal{C} \times_3 \mathbf{B}$. *Let* $L(\omega)$ *be the loss at frequency* ω. *For any* $\omega_{high} > \omega_{low} > 0$ *and any* $\epsilon \geq 0$, *there exists a matrix* $\mathbf{B}$ *such that for all* s*:*

$$\left| \frac{\partial L(\omega_{high})}{\partial \mathcal{C}_{pqs}} \bigg/ \frac{\partial L(\omega_{low})}{\partial \mathcal{C}_{pqs}} \right| \geq \max_j \left| \frac{\partial L(\omega_{high})}{\partial \mathcal{G}_{pqj}} \bigg/ \frac{\partial L(\omega_{low})}{\partial \mathcal{G}_{pqj}} \right| - \epsilon.$$

Insight: The reparameterization inherently **amplifies the gradient response** to high-frequency components relative to low-frequency ones, enhancing optimization in high-frequency directions.

Basis Initialization

We adopt a **Xavier-style initialization** for the entries of $\mathbf{B}^{(k)}$:

Theorem 3 (Basis Initialization)

Suppose the entries of $\mathbf{B}^{(k)}$ *in the reparameterized TR factor are sampled independently from a uniform distribution as*

$$\mathbf{B}_{ij}^{(k)} \sim \mathcal{U}\left(-\sqrt{\frac{6}{r_{k+1}+R_{k+1}}}, \sqrt{\frac{6}{r_{k+1}+R_{k+1}}}\right), \forall i, j, k,$$

then the variances in both the forward and backward passes are preserved.

Lipschitz Continuity

RepTRFD satisfies the **Lipschitz continuity** property:

Theorem 4 (Lipschitz Continuity)

For the proposed model $g_\phi(\mathbf{v}) = \Phi\left(\{g_{\phi_k}(v_k) \times_3 \mathbf{B}^{(k)}\}_{k=1}^{d}\right)$, *assume for all modes* k*:*

- *MLPs* g_{ϕ_k} *have* κ*-Lipschitz activations and bounded weight norms* ($\leq \eta$).
- *MLP outputs are bounded:* $\sup_{v_k} \|g_{\phi_k}(v_k)\|_F \leq C_k$.

Then, $g_\phi(\mathbf{v})$ *is* **globally Lipschitz continuous***:*

$$|g_\phi(\mathbf{v}) - g_\phi(\mathbf{v}')| \leq \delta \|\mathbf{v} - \mathbf{v}'\|_2,$$

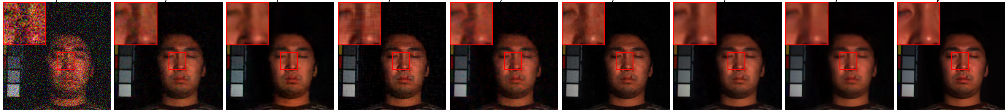
with the Lipschitz constant bounded by $\delta = \sqrt{\sum_{k=1}^{d} \delta_k^2}$, *where* $\delta_k = (\kappa\eta)^{L_k} \|\mathbf{B}^{(k)}\|_2 \prod_{j \neq k} C_j$.

Experimental Results

1. Inpainting Results:

Observed	TRLRF	FCTN	HLRTF	LRTFR	DRO-TFF	NeurTV	Ours	Ground truth
PSNR/SSIM	26.76/0.812	27.74/0.837	29.64/0.914	29.62/0.893	30.66/0.937	30.05/0.902	**32.55/0.957**	**Inf/1.000**

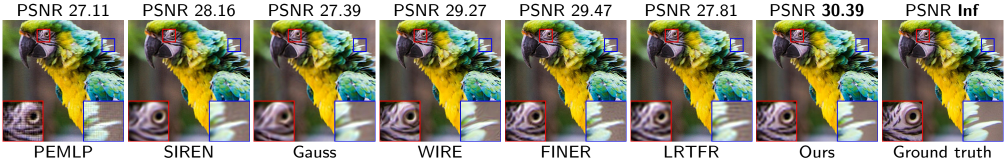
2. Denoising Results:



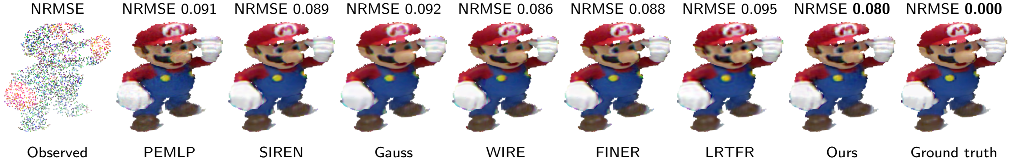
Observed	LRTDTV	DeepTensor	HLRTF	LRTFR	DRO-TFF	NeurTV	Ours	Ground truth
PSNR/SSIM	32.26/0.701	34.06/0.859	34.67/0.835	35.13/0.858	36.37/0.920	36.78/0.927	**37.89/0.945**	**Inf/1.000**

RepTRFD achieves **more faithful reconstructions** and consistently higher PSNR and SSIM.

3. Super-Resolution Results:



4. Point Cloud Recovery Results:



RepTRFD exhibits **stronger generalization capability** and **improved continuous modeling ability**.

Effect of Reparameterization

- **Improved Training Dynamics:** Models with reparameterization **converge faster**. Larger β accelerates convergence.
- **Better Performance-Cost Trade-off:** The reparameterization strategy substantially **enhances reconstruction quality** with **marginal computational cost**.

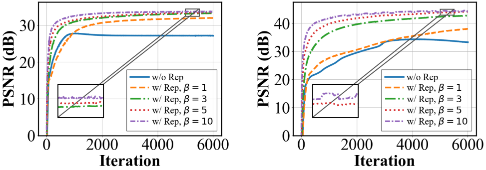


Figure: PSNR curves comparing w/o Rep. and w/ Rep. with different β values.

Table: Effect of reparameterization on PSNR and runtime for different datasets.

Metric	*Airplane*		*Toy*		*Botswana*		*News*	
	w/o Rep.	w/ Rep.	w/o Rep.	w/ Rep.	w/o Rep.	w/ Rep.	w/o Rep.	w/ Rep.
PSNR	27.41	**30.45**	29.41	**48.67**	29.21	**45.27**	26.48	**34.90**
Time (s)	12.10	13.11	13.23	14.49	40.81	42.26	14.15	15.59

- **Sensitivity of Basis Initialization:** The theoretically derived Xavier initialization scale achieves **optimal performance**.
- **Scalability to Higher-Order Tensors:** RepTRFD **scales well** to higher-order data.

Table: PSNR under different initialization scales a for $\mathcal{U}(-a, a)$.

Data	Initialization Scale a					
	0.01	0.05	0.165 (ours)	0.3	0.5	1
Botswana	33.86	43.74	**45.27**	44.81	43.15	38.37
Washington DC	33.14	46.08	**47.96**	47.44	45.90	38.20

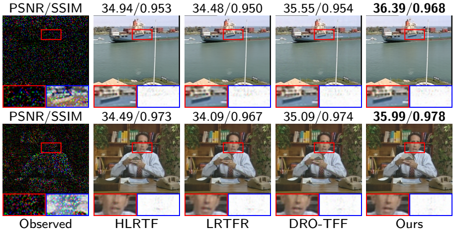


Figure: Color video inpainting results.

Summary & Future Work

- **Continuous Extension:** Extended TR decomposition to the **continuous domain** and provided a **frequency-domain analysis** of TR factors.
- **Reparameterization Strategy:** Proposed TR factor reparameterization for **improved high-frequency learning**.
- **Theoretical Guarantees:** Proved **enhanced training dynamics**, derived a variance-preserving **initialization**, and established global **Lipschitz continuity**.
- **Future Work:** Apply the proposed reparameterization technique to other tensor decomposition frameworks.

Article **Code**

Thank you!